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STATISTICAL-KINEMATIC ANALYSIS AND LIMIT EQUILIBRIUM OF SYSTEMS WITH UNILATERAL CONSTRAINTS

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Abstract—A general mechanical system involving unilateral constraints is studied and contrasted to a system with bilateral constraints. Some far-reaching consequences of their fundamental dissimilarity are revealed and explained. In particular, it is demonstrated that in the course of loading, a system with unilateral constraints can convert into a stationary (non-working) plastic mechanism. In this case the conventional version of the limit equilibrium method will miss the corresponding value of the load parameter. Therefore, certain precautions must be taken when applying the method, especially, in computer-assisted analysis.

INTRODUCTION

Structural elements capable of sustaining an external force of only one sign are represented in a design model by unilateral constraints. Systems involving unilateral constraints quite often do not belong to geometrically stable structures; they are, rather, kinematic chains with many degrees of freedom. Yet some of these systems, when made of an undeformable material, completely lack mobility. Such peculiar systems were first envisioned by O. Mohr in 1885 and J. Maxwell in 1890. However, only in 1930 T. Levi-Civita and U. Amaldi[1] revealed and explained analytically the essence of the phenomenon. Now these systems are widely used in practice (suspended roofs, so-called "tensegrity" structures[2–4], etc.).

Verification of geometrical stability or otherwise, as well as determination of other basic properties of a mechanical system is the subject of statical-kinematic analysis. The latter deals with all properties of a system which are irrelative to its material. Since the limit design of rigid-ideal-plastic systems does not involve stress–strain relationships it also falls within the scope of statical-kinematic analysis.

The objective of this paper is an extension of statical-kinematic analysis (especially, limit design) in two ways: 1. To systems other than geometrically stable; 2. To systems involving unilateral constraints.

Obviously, these two categories overlap; however, when combined, they provide the extension of statical-kinematic analysis to a general mechanical system.

BASIC CONCEPTS AND DEFINITIONS

Within the framework of statical-kinematic analysis a mechanical system is taken to mean an assemblage of rigid bodies linked by ideal positional bilateral and/or unilateral constraints. The two types of constraints regulate the relative positions of the members by fixing or limiting, respectively, the distances, angles and other geometrical parameters of the system. Accordingly, the constraint conditions take the analytical form of equations and inequalities:

$$F_i(X_1, \ldots, X_n, \ldots, X_N) = 0, \quad i = 1, 2, \ldots, I. \qquad (1)$$

$$n = 1, 2, \ldots, N$$

$$F_j(X_1, \ldots, X_n, \ldots, X_N) \leq 0, \quad j = 1, 2, \ldots, J. \qquad (2)$$

Here X_n are the generalized coordinates, N in number, uniquely determining the configuration of the system, i.e. the simultaneous position of each of its material points while I and J is the number of bilateral and unilateral constraints, respectively. The geometrical parameters included in the arguments of F_i and F_j are not specified explicitly. Some of the F_i and F_j may be linearly dependent functions, and the corresponding constraints are then said to be dependent. The set of equations and inequalities (1–2) is assumed to be consistent—that is, to have at least one real solution $X_n = X_n^0$.

As is known[1], linearization of the set (1–2), and subsequent use of the virtual displacement principle, yield: (a) the constraint conditions in differential form which specify restrictions on the infinitesimal increments of the generalized coordinates (the virtual displacements of the system), and (b) the homogeneous conditions of equilibrium with regard to each generalized coordinate:

$$f_{in}x_n = 0, \quad f_{jn}x_n \leq 0 \tag{3}$$

$$f_{in}\lambda_i + f_{jn}\mu_j = 0, \quad \mu_j \geq 0. \tag{4}$$

In expressions (3) and (4)

$$f_{in} = \left.\frac{\partial F_i}{\partial X_n}\right|_0, \quad f_{jn} = \left.\frac{\partial F_j}{\partial X_n}\right|_0, \tag{5}$$

i.e. the derivatives are evaluated at a fixed point $X_n = X_n^0$ of the coordinate space of the system. The Lagrange multipliers λ_i and μ_j are the generalized reactions of the respective constraints; x_n are the virtual displacements; a repeated (dummy) index is used to denote summation over all its values.

It should be noted that the only non-zero coefficients f_{jn} are the ones associated with engaged unilateral constraints, that is with those for which relations (2) hold as equalities in the considered configuration of the system. Only engaged constraints affect the infinitesimal virtual displacements in the given configuration; for the disengaged constraints, the corresponding relations take the form of strict inequalities and their presence is not reflected in equalibrium and linearized constraint conditions. Obviously, bilateral constraints are always engaged, whereas engagement or otherwise of unilateral constraints depends on the configuration of the system. Note that engagement only enables but does not necessitate the unilateral constraint to develop a reaction.

The relations written down contain all the information on the statical-kinematic properties of a mechanical system. Their more detailed analysis[2, 3] showed that there exist four and only four statical-kinematic types of mechanical systems. Proceeding from uniqueness (or otherwise) of the configuration of a system and its capability (or otherwise) of balancing external loads, the definitions of these four types are as follows.

Definitions. A (geometrically) stable system is one possessing a unique configuration and capable of balancing an arbitrary finite load under finite reactions of its constraints.

A quasi-stable system is one possessing a unique configuration, incapable of balancing an arbitrary load, but permitting such a variation of constraints† which renders it stable.

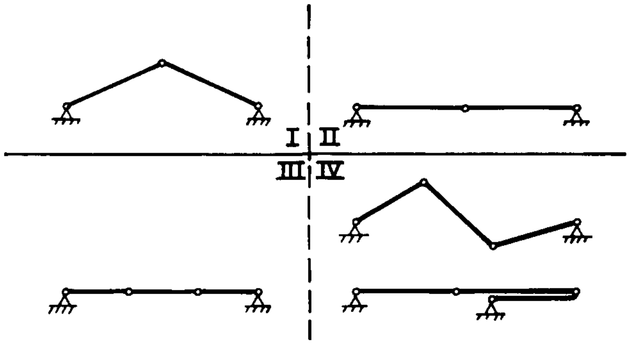
A quasi-unstable system is one possessing a unique configuration, incapable of balancing an arbitrary load, and not permitting such a variation of constraints, which would render it stable.

An unstable system is one possessing a nonunique configuration, i.e. permitting a variety of configurations under trival (zero) variation of constraints.

On this basis, the statical-kinematic classification of mechanical systems may be presented in the form of a four-quadrant table with the main diagonal (quadrants I and IV) containing the main types—stable and unstable—and the secondary diagonal (quadrants II and III)—the degenerate types, quasi-stable and quasi-unstable. Tables 1–2 present the simplest prototypes of the four statical-kinematic types of systems consisting of no more than three bars (bilateral constraints) or wires (unilateral constraints), respectively. Both quasi-stable and quasi-unstable systems have a positive number of degrees of freedom, but nevertheless completely lack kinematic (as distinct from elastic) mobility. The uniqueness of their configurations is the consequence of their degeneracy which results in two characteristic features. Firstly, it yields the possibility of infinitesimal displacements under infinitesimal second or higher order deformation of constraints. Secondly, a general variation of constraints brings a quasi-stable or quasi-unstable system out of the degenerate configuration thus rendering it either stable or

†More exactly, the variation is in the geometric parameters figuring in the constraint conditions. The variation can be caused by elastic, plastic, temperature, and other deformations of the real constraints or by a change in the parameters of the analytical model.

Table 1.



unstable, respectively (see definitions).

Note that "tensegrity" systems are quasi-unstable.

ANALYSIS OF SYSTEMS INVOLVING UNILATERAL CONSTRAINTS

To analyze a system involving unilateral constraints, consider a more general case where the bilateral constraints alone are insufficient to provide geometrical stability of the system. In this case the rank r_i of the matrix of coefficients f_{in} is less than the number of degrees of freedom: $r_i < N$. Therefore, some r_i of the displacements x_k in eqns (3) can be expressed in terms of the remaining $(P = N - r_i)$ displacements x_p taken as independent:

$$x_k = a_{kp}x_p, \quad p = 1, 2, \ldots, P, \quad k = P+1, \quad P+2, \ldots, N. \tag{6}$$

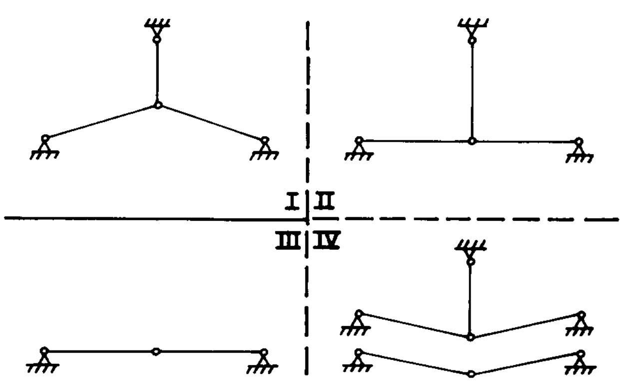
Substituting these expressions into inequalities (3) yeilds

$$(f_{jp} + f_{jk}a_{kp})x_p \equiv c_{jp}x_p \leq 0, \quad j = 1, 2, \ldots, J, \quad p = 1, 2, \ldots, P. \tag{7}$$

Now the constraint conditions are expressed in terms of the independent displacements x_p of the system taken without its unilateral constraints. The necessary and sufficient analytical criterion for a system to be stable is the existence of only the trivial solution $x_p = 0$ for the set of linear inequalities (7). Then all x_k will also be zero and no displacements are possible (uniqueness of the configuration). To satisfy this requirements, the rank r_j of the matrix $C = c_{jp}$ given by (7) must be equal to the number P of displacements and the number J_0 of independent inequalities (otherwise, even the corresponding boundary set of equations would admit a non-zero solution), while the total number J of inequalities must be $J > J_0$. Supposing these conditions to be satisfied for set (7), let one of its basic subsets (i.e. a subset corresponding to one of the principal minors C_0 of matrix (C) be

$$c_{qp}x_p \leq 0, \quad q, p = 1, 2, \ldots, P \tag{8}$$

Table 2.



In the coordinate space of x_p, this subset determines the domain (a convex polyhedral cone) of admissible values of the coordinates for a sybsystem whose constraints are those represented in the minor C_0. Suppose, that among the inequalities not covered by C_0, there exists (or can be constructed from them as a linear combination with positive coefficients d_m) an inequality

$$c_{op}x_p(\equiv d_m c_{mp} x_p) \leq 0, \quad m = P+1, P+2, \ldots, J \tag{9}$$

such that the above cone (except for its apex) falls outside the half-space defined by set (9). In this case subset (8) on the one hand, and the inequality (or group of inequalities) not covered by it on the other, are mutually exclusive, a circumstance which rules out the above cone as a domain of admissible values of the coordinates, thus reducing the domain to a single point $x_p = 0$. Obviously, the existence in set (7) of a basic subset and an inequality, having the described property, is necessary and sufficient for the absence of nonzero solutions for this set. The condition under which this takes place is well known in the theory of linear inequalities [5] and requires the existence of a group of positive numbers μ_s satisfying the set of homogeneous equations

$$c_{sp}\mu_s = 0, \quad s = 0, 1, 2, \ldots, P. \tag{10}$$

Equations (10) are nothing but the homogeneous equilibrium equations in the unknown μ_s which are the generalized reactions of the unilateral constraints. The possibility of having nonzero reactions in the absence of external loads means that the system admits initial forces and is statically indeterminate. Hence, it becomes clear why a geometrically stable system involving even one essential unilateral constraint is statically indeterminate. Unlike a system with bilateral constraints, simultaneous statical and kinematic determinacy is precluded for a system with unilateral constraints: a stable (i.e. kinematically determinate) system is statically indeterminate and conversely, a statically determinate system (i.e. one not admitting initial forces) is not geometrically stable. In other words, the alterability of a system with unilateral constraints can be eliminated only by introducing counteractive constraints of the type (9) which (by definition) are dependent and therefore give rise to statical indeterminacy.

As follows from the above, the degree of statical indeterminacy for a general mechanical system is given by

$$n_s = I + J - (r_i + r_j) \tag{11}$$

that is the total number of constraints essential for the configuration less the sum of the ranks of matrices (5). The degree of kinematical indeterminacy equals the dimensionality of the solution cone of set (7):

$$n_k = r_c. \tag{12}$$

LIMIT EQUILIBRIUM—CURRENT AND EXTENDED CONCEPTS

Like many other concepts of statical-kinematic analysis, the basic objects of the limit equilibrium theory are of a local character, i.e. pertinent to a fixed configuration of a mechanical system rather than to the system *per se*. It is known, for example, that in a geometrically non-linear system, the limit load found for its initial configuration may be either greater or less than the actual load-carrying capacity of the system; the first is the case (in particular) with convex arches and shells, the second, with suspended systems.

The traditional subject of the theory of limit equilibrium is a geometrically stable system. It can, however, be extended to a general system carrying an equilibrium load,† in the following way. If an equilibrium load on a system increases proportionally to a parameter, any displacements must be due exclusively to the flexibility of the constraints. Given their undeformability, the configuration of the system is fixed regardless of whether or not its composition admits

†An equilibrium load is one which a system in a given configuration is capable of balancing by finite internal forces. Obviously, for a geometrically stable system, any finite load is an equilibrium load.

displacement without deformation (kinematic displacement). Thus, from the viewpoint of equilibrium any balanced system does not differ from a stable one. This is stated more rigorously in the following.

Lemma. If a system is in equilibrium, imposition of an independent constraint changes neither the number nor the form of the statically-possible states corresponding to the given load (the introduced constraint is idle).

Proof. Imposition of an independent constraint deprives the system of one degree of freedom. The release of such a constraint will not affect the equilibrium provided its reaction is applied instead. In the absence of this reaction, the work of all the external and internal forces over any virtual displacement is zero—as the system is in equilibrium according to the condition of the lemma. This means that the virtual work of the reaction is also zero, i.e. under the given load the constraint is idle, which was to be proved.

Any system can be rendered stable by imposition of an appropriate number of independent constraints. The fact that according to the lemma this operation does not affect the equilibrium, applies to the limit equilibrium as well. This ensures applicability of all existing theorems and of both methods (kinematic and statical) of the theory of limit equilibrium, irrespective of the type of system, and leads to the conclusion that the theory of limit equilibrium is equally valid for all systems, of whatever statical-kinematic type.

However, extension of this theory to an arbitrary system calls for certain refinements. In accordance with the current approach, the state of limit equilibrium sets in at the moment when the system becomes a plastic mechanism. The drawback of this criterion—even when applied to a geometrically stable system—is that the limit equilibrium, a purely statical concept, is determined in kinematic terms. In a system of any other type, possessing from the outset a positive number of degrees of freedom, this criterion clearly needs refinement in order to restore its statical origin; this entails no difficulty.

In statical terms, the occurrence of the limit equilibrium means that, in the course of loading, the system (namely, its constraints) has reached a state at which the external load can no longer be balanced in a given configuration.

Definition. The (one-parametric) limit load is one corresponding to that value of its parameter which, on being reached, renders the equilibrium in a given configuration impossible. The limit equilibrium is that corresponding to the limit load.

This definition, naturally referring to all types of systems, is not totally identical with the conventional criterion of limit equilibrium of a stable system (conversion into a plastic mechanism). The fact is that such a conversion does not necessarily result in the onset of motion under further loading; as will be demonstrated, equilibrium in the same configuration can still be possible. Specifically, a geometrically stable system with unilateral constraints can convert into a stationary (non-working) plastic mechanism, i.e. one for which the acting load continues to be balanced. This situation promptly raises a number of problems under the conventional approach.

First of all, it is not always justified to regard such a state as the limit equilibrium, because under certain conditions the load may be further increased until the equilibrium becomes unstable and the system is converted into a working plastic mechanism. This depends on the condition—stable or otherwise— of the equilibrium of the first, nonworking, plastic mechanism. In the case of stability, the capacity for increased loading is quite real and represents the reserve carrying capacity of the system. By contrast, the case of instability entails the risk of an over-estimated carrying capacity: since the plastic mechanism is balanced and therefore stationary, the corresponding parameter of the load can not be obtained by equating the virtual work of the external and internal forces. The conventional procedure, based on examination of all kinematically possible patterns, will miss this value of the parameter and lead to a higher value corresponding to some other pattern—one with a working plastic mechanism.

Although the above definition does not obviate these difficulties, it contributes to their proper recognition. As for the problem itself, it can be demonstrated that the situation in question is confined to systems involving unilateral constraints.

Proposition. Conversion of a system with bilateral constraints into a balanced kinematic chain (mechanism) with a plastic constraint is impossible.

Proof. Conversion of a system with bilateral constraints into a plastic mechanism can only result from the yielding of an independent constraint. However, according to the lemma, if the resulting mechanism is in equilibrium, imposition of an independent constraint in place of the yielded constraint would not affect the force distribution under the given load. Therefore, the reaction of this constraint could not increase during loading, at least once it has become independent (on yielding of some other constraints). Consequently, such a constraint could not reach yielding, thus ruling out the possibility of formation of a balanced plastic mechanism.

LIMIT EQUILIBRIUM OF SYSTEMS WITH UNILATERAL CONSTRAINTS

As regards a system with unilateral constraints, a plastic mechanism (in particular, a balanced one) may also form through yielding of a dependent constraint which can even be bilateral; the only important thing is that it be counteractive.

As can be seen from formula (11) yielding of a dependent constraint always decreases the degree of statical indeterminacy by one. However, if the constraint was also counteractive, the system can acquire one or more degrees of freedom depending on the subsystem which the constraint was counteractive to. This specific feature constitutes the difference of principle between a system with bilateral constraints and a system involving unilateral constraints.

The following example illustrates the consequences of constraint yielding and the peculiarities of limit equilibrium of systems with unilateral constraints. Let the beam of the system shown in Fig. 1 be capable of resisting a bending moment of only one sign (positive). This geometrically stable system is singly statically indeterminate; let the stiffness ratio of its members be such that the beam yields first over one of the posts. This suffices for the system to convert into a plastic mechanism (Fig. 2). However, the given load is the equilibrium load for the formed mechanism: the total work (of first order) of the external forces is zero for its single (antisymmetric) virtual displacement. That is, this is a stationary plastic mechanism.

Before investigating all the consequences of the above phenomenon, it is worthwhile to use this example to show why, as claimed by the proposition proved before, this is impossible for a system with bilateral constraints. The reason is that in the case of a beam capable of resisting a bending moment of either sign the system will remain geometrically stable after formation of the first hinge, and the cross section of the beam over the second post becomes an independent constraint. Since the external load is an equilibrium load for the subsystem not involving this independent constraint, the force in the latter will not increase on further loading. In other words, the bending moment in this cross section will retain its former value, precluding formation of plastic hinge. This will continue until yielding sets in some other constraint; as all of them are now independent, yielding of any of them converts the system into a plastic mechanism, nesessarily a working (moving) one.

Thus, the difference between the two variants is that at the instant of formation of the plastic hinge, the system in the first case (unilateral constraints) turns into a stationary (balanced) plastic mechanism, while in the second case, it looses its statical indeterminacy but remains geometrically stable. In both cases, the resulting state is not that of limit equilibrium; the corresponding load cannot be found by the known methods of the theory of limit equilibrium; the load-carrying capacity is not exhausted, and further loading is possible. Obviously, in the first case, the system becomes sensitive to any asymmetrical action (e.g. inequality of the vertical forces).

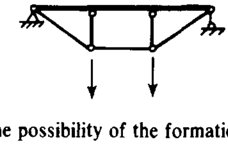


Fig. 1. Structure demonstrating the possibility of the formation of a balanced plastic mechanism.

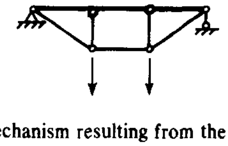


Fig. 2. Stationary plastic mechanism resulting from the yield of one of the constraints.

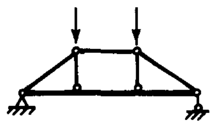


Fig. 3. Structure which converts into an unstable balanced plastic mechanism.

The strength reserve can be considerable in both cases, and the limit equilibrium theory permits determination of the load-carrying capacity of both systems. The fact that these methods cannot predict conversion of the system into a plastic mechanism in the first case, is not necessarily dangerous, provided stability is unsured on further loading. However, under unilateral constraints, a somewhat more dangerous situation may arise. For the system of Fig. 3, the process of loading is similar to that for its predecessor, but after formation of a plastic hinge, the equilibrium condition may prove kinematically unstable before yielding sets in the next constraint. The difficulty is that for not too trivial systems with unilateral constraints, the onset of such a state can only be detected by incremental elasto–plastic analysis with stepwise examination of stability. This clearly falls beyond the scope of the classical limit equilibrium theory.

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